

## Note

# Re-examination of selection index for desired gains

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## Summary

The method of index selection for desired genetic changes derived by BRASCAMP (1979) was re-examined and we found that his method gave the equivalent solutions to that of YAMADA *et al.* (1975). Brascamp's method is more complicated and has no advantage compared to the method of YAMADA *et al.*

*Key words* : Selection index, desired gains, constrained selection index.

## Résumé

*Nouvel examen des index de sélection pour des gains génétiques espérés*

Nous avons réexaminé la méthode de sélection par index pour des gains génétiques espérés proposée par BRASCAMP (1979) et nous trouvons que sa méthode fournit des solutions équivalentes à celles de YAMADA *et al.* (1975). La méthode de Brascamp est plus compliquée et ne présente pas d'avantage par rapport à celle de YAMADA *et al.*

*Mots clés* : Index de sélection, gains espérés, index de sélection sous contrainte.

A selection index for desired genetic changes was derived by PESEK & BAKER (1969). They restricted themselves to the situations where the traits in the index are exactly the same as those in the aggregate genotype. YAMADA *et al.* (1975) gave a more general solution. Their method is available even in situations where the index contains traits not included in the aggregate genotype. Similar results have been obtained by HARVILLE (1975), ROUVIER (1977), ESSL (1981) and TALLIS (1985). The most different point of YAMADA *et al.* (1975) from the others was that they did not assume any economic weight nor underlying aggregate genotypic value.

BRASCAMP (1979) discussed another solution of this problem and a detailed derivation of his method was written in his review paper (BRASCAMP, 1984). His method

appears quite different from that of YAMADA *et al.* (1975), but we found that they were exactly the same. Because the method of YAMADA *et al.* is simpler than that of BRASCAMP, we can say that the former is preferable to the latter.

The objective of this note is to prove their equivalence algebraically.

To describe the selection indices we use the following notations.  $\mathbf{b}$  is an  $n \times 1$  vector of weighting factors.  $\mathbf{p}$  is an  $n \times 1$  vector of phenotypic values of individuals as deviations from their relevant means.  $\mathbf{g}$  is an  $m \times 1$  vector of additive genetic merits.  $\mathbf{d}$  is an  $m \times 1$  vector of desired relative genetic changes. The variance covariance matrix of  $\mathbf{p}$  is denoted as  $\mathbf{P}$  with order  $n \times n$ . The covariance matrix between  $\mathbf{p}$  and  $\mathbf{g}$  is denoted as  $\mathbf{G}$  with order  $n \times m$ , *i.e.*  $\text{Cov}(\mathbf{p}, \mathbf{g}) = \mathbf{G}$ . We do not assume any economic weight, nor aggregate genotypic value.

First we describe the selection index of YAMADA *et al.* (1975). The expected gains after applying the selection index  $I = \mathbf{b}'\mathbf{p}$  in one generation are :

$$E(\Delta\mathbf{g}) = i\mathbf{G}'\mathbf{b}/\sigma_I$$

where  $i$  is the intensity of selection and  $\sigma_I$  is the standard deviation of the index, *i.e.*  $\sigma_I = \sqrt{\mathbf{b}'\mathbf{P}\mathbf{b}}$ . Because  $E(\Delta\mathbf{g})$  is proportional to  $i$  and  $\sigma_I$ , it is sufficient to solve  $\mathbf{b}$  such that :

$$\mathbf{G}'\mathbf{b} = \mathbf{d} \quad (1)$$

then  $E(\Delta\mathbf{g}) = i\mathbf{d}/\sigma_I$ . Now suppose  $n > m$  and  $\mathbf{G}$  has full column rank. If  $n < m$  or  $\mathbf{G}$  is not of full column rank, equations (1) may be inconsistent and have no solution in general. If these equations are consistent, they have solutions, but no unique solution exists. Now  $E\Delta\mathbf{g}$  is inversely proportional to  $\sigma_I$ , so that the best choice among all solutions is  $\mathbf{b}$  causing  $\sigma_I$  to be minimum subject to the constraints (1), because then  $E(\Delta\mathbf{g})$  is maximum. Such a solution can be found by putting partial derivatives of :

$$\frac{1}{2} \mathbf{b}'\mathbf{P}\mathbf{b} + \boldsymbol{\lambda}'(\mathbf{G}'\mathbf{b} - \mathbf{d})$$

with respect to  $\mathbf{b}$  and  $\boldsymbol{\lambda}$  to zeros, where  $\boldsymbol{\lambda}$  is an  $m \times 1$  vector of Lagrange multipliers. Then we get the equations :

$$\begin{bmatrix} \mathbf{P} & \mathbf{G} \\ \mathbf{G}' & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{b} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{d} \end{bmatrix} \quad (2)$$

and solving these equations as to  $\mathbf{b}$ , we finally get :

$$\mathbf{b} = \mathbf{P}^{-1}\mathbf{G}(\mathbf{G}'\mathbf{P}^{-1}\mathbf{G})^{-1}\mathbf{d} \quad (3)$$

which is the selection index derived by YAMADA *et al.* (1975).

This result is also derived easily from the fact that  $\mathbf{P}^{-1}\mathbf{G}(\mathbf{G}'\mathbf{P}^{-1}\mathbf{G})^{-1}$  is a minimum norm generalized inverse of  $\mathbf{G}'$ .

BRASCAMP (1979) modified the equations (1) as :

$$\mathbf{G}\mathbf{G}'\mathbf{b} = \mathbf{G}\mathbf{d} \quad (4)$$

by pre-multiplying  $\mathbf{G}$ . Because  $\mathbf{GG}'$  is singular, no unique solution for  $\mathbf{b}$  exists, and all solutions can be denoted by :

$$\mathbf{b} = (\mathbf{GG}')^{-1}\mathbf{Gd} + [(\mathbf{GG}')^{-1}\mathbf{GG}' - \mathbf{I}]\mathbf{z} = \mathbf{Ad} + \mathbf{Bz} \quad (5)$$

where  $\mathbf{A} = (\mathbf{GG}')^{-1}\mathbf{G}$ ,  $\mathbf{B} = (\mathbf{GG}')^{-1}\mathbf{GG}' - \mathbf{I}$ ,  $(\mathbf{GG}')^{-1}$  is a generalized inverse of  $\mathbf{GG}'$  and  $\mathbf{z}$  is an arbitrary  $n \times 1$  vector. The best choice among all solutions is  $\mathbf{b}$  causing the variance of the index to be minimum. It can be shown that  $\mathbf{z} = -(\mathbf{B}'\mathbf{PB})^{-1}\mathbf{B}'\mathbf{PAd}$  minimizes  $\mathbf{b}'\mathbf{Pb}$ . Substituting this  $\mathbf{z}$  into (5), we get :

$$\mathbf{b} = [\mathbf{I} - \mathbf{B}(\mathbf{B}'\mathbf{PB})^{-1}\mathbf{B}'\mathbf{P}]\mathbf{Ad} \quad (6)$$

The resulting solution is invariant to the choice of  $(\mathbf{B}'\mathbf{PB})^{-1}$  as proved by BRASCAMP (1984).

The method of BRASCAMP looks quite reasonable, but it has an improper point shown in the following. The equations (4) look like normal equations used in the least squares procedure. That is, it seems as if he applied the least squares procedure to (1) and get the normal equations (4). The equations (1), however, are consistent, so they hold good without error and it is obvious that their solutions are exactly the same as those of the equations (4). Further both of the method given by YAMADA *et al.* and BRASCAMP adopt  $\mathbf{b}$  which minimizes the variance of the index. These facts can make us understand intuitively the equivalence of these 2 methods.

Now we can prove the equivalence of these methods algebraically. We must use the following lemma.

LEMMA 1. Let  $\mathbf{X}_{p \times r}$  and  $\mathbf{K}_{p \times (p-q)}$  be of rank  $q$  and  $(p - q)$  such that  $\mathbf{K}'\mathbf{X} = \mathbf{0}$ . Then if  $\mathbf{V}_{p \times p}$  is a symmetric positive definite matrix, then :

$$\mathbf{V}^{-1} - \mathbf{V}^{-1}\mathbf{X}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1} = \mathbf{K}(\mathbf{K}'\mathbf{VK})^{-1}\mathbf{K}'$$

The proof of this lemma is indicated in an appendix.

$\mathbf{B}$  has order  $n \times n$  and rank  $(n - m)$ ,  $\mathbf{G}$  has order  $n \times m$  and rank  $m$ ,  $\mathbf{P}$  is symmetric positive definite, and further :

$$\mathbf{G}'\mathbf{B} = \mathbf{G}'(\mathbf{GG}')^{-1}\mathbf{GG}' - \mathbf{G}' = \mathbf{G}' - \mathbf{G}' = \mathbf{0},$$

so that, using lemma 1, it can be shown that :

$$\mathbf{P} - \mathbf{PB}(\mathbf{B}'\mathbf{PB})^{-1}\mathbf{B}'\mathbf{P} = \mathbf{G}(\mathbf{G}'\mathbf{P}^{-1}\mathbf{G})^{-1}\mathbf{G}'$$

Using this, BRASCAMP's selection index can be rewritten as :

$$\begin{aligned} \mathbf{b} &= [\mathbf{I} - \mathbf{B}(\mathbf{B}'\mathbf{PB})^{-1}\mathbf{B}'\mathbf{P}]\mathbf{Ad} \\ &= \mathbf{P}^{-1}[\mathbf{P} - \mathbf{PB}(\mathbf{B}'\mathbf{PB})^{-1}\mathbf{B}'\mathbf{P}]\mathbf{Ad} \\ &= \mathbf{P}^{-1}\mathbf{G}(\mathbf{G}'\mathbf{P}^{-1}\mathbf{G})^{-1}\mathbf{G}'\mathbf{Ad} \end{aligned} \quad (7)$$

Because  $\mathbf{G}$  has full column rank, it can be partitioned as :

$$\mathbf{G} = \begin{bmatrix} \mathbf{G}_1 \\ \mathbf{G}_2 \end{bmatrix}$$

where  $\mathbf{G}_1$  is an  $m \times m$  non-singular matrix and  $\mathbf{G}_2$  is an  $(n - m) \times m$  matrix. We use the following generalized inverse of  $\mathbf{GG}'$  for simplification.

$$(\mathbf{GG}')^- = \begin{bmatrix} (\mathbf{G}_1\mathbf{G}_1')^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

Then we get :

$$\mathbf{G}'\mathbf{A} = \mathbf{G}'(\mathbf{GG}')^- \mathbf{G} = \mathbf{G}'(\mathbf{G}_1\mathbf{G}_1')^{-1}\mathbf{G}_1 = \mathbf{I}$$

This is always true for an arbitrary generalized inverse of  $\mathbf{GG}'$ , because  $\mathbf{G}'(\mathbf{GG}')^- \mathbf{G}$  is invariant to the choice of the generalized inverse as well known. Using this result, (7) becomes :

$$\begin{aligned} \mathbf{b} &= \mathbf{P}^{-1}\mathbf{G}(\mathbf{G}'\mathbf{P}^{-1}\mathbf{G})^{-1}\mathbf{G}'\mathbf{A}\mathbf{d} \\ &= \mathbf{P}^{-1}\mathbf{G}(\mathbf{G}'\mathbf{P}^{-1}\mathbf{G})^{-1}\mathbf{d} \end{aligned}$$

This formula is exactly the selection index of YAMADA *et al.* represented by (3).

As mentioned above, these 2 methods of YAMADA *et al.* and BRASCAMP give equivalent solutions. Furthermore, BRASCAMP's method is more complicated than that of YAMADA *et al.* Therefore, BRASCAMP's method has no advantage compared to that of YAMADA *et al.*

### Numerical Example

We use the same example used by BRASCAMP (1979) as follows.

$$\mathbf{P} = \begin{bmatrix} 100.00 & 30.00 & 15.00 \\ & 36.00 & 18.00 \\ & & 25.00 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 25.00 & 9.00 \\ 9.00 & 12.96 \\ 6.00 & 8.64 \end{bmatrix},$$

$$\mathbf{d} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

We can use the 3 different formulae (2), (3) and (6) to compute the selection index, but whichever formula we may use, we get the identical solution :

$$\mathbf{b} = \begin{bmatrix} -0.020741 \\ 0.133091 \\ 0.053450 \end{bmatrix}$$

Then the variance of the selection index is :

$$\sigma_i^2 = \mathbf{b}'\mathbf{P}\mathbf{b} = 0.809327$$

The expected genetic gains in one generation of selection are :

$$E(\Delta\mathbf{g}) = \frac{i\mathbf{d}}{\sigma_i} = i \begin{bmatrix} 1.111573 \\ 2.223146 \end{bmatrix}$$

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### Appendix

#### *Proof of lemma 1*

Let us partition  $\mathbf{X}$  as  $\mathbf{X} = [\mathbf{W} : \mathbf{WF}]$ , where  $\mathbf{W}$  is of order  $p \times q$  and full column rank and  $\mathbf{F}$  is of order  $q \times (r - q)$ . Because  $\mathbf{K}'\mathbf{X} = \mathbf{0}$ ,  $\mathbf{W}$  also satisfies the condition  $\mathbf{K}'\mathbf{W} = \mathbf{0}$ . Because  $\mathbf{V}$  is symmetric positive definite, there exists a non-singular matrix

$T_{p \times p}$  such that  $V = TT'$ . Similarly let  $(W'V^{-1}W)^{-1} = QQ'$  and  $(K'VK)^{-1} = RR'$  where  $Q_{q \times q}$  and  $R_{(p-q) \times (p-q)}$  are non-singular matrices. We define  $S_{p \times p}$  as  $S = [T^{-1}WQ : T'KR]$ . Then  $S$  has full rank and  $S'S = I_p$ , so that  $S$  is an orthogonal matrix. Therefore :

$$I_p = SS' = T^{-1}WQQ'W'T'^{-1} + T'KRR'K'T,$$

then :

$$I - T^{-1}W(W'V^{-1}W)^{-1}W'T'^{-1} = T'K(K'VK)^{-1}K'T$$

Pre-multiplying  $T'^{-1}$  and post-multiplying  $T^{-1}$ , we get :

$$V^{-1} - V^{-1}W(W'V^{-1}W)^{-1}W'V^{-1} = K(K'VK)^{-1}K' \quad (A1)$$

This is the result derived by KHATRI (1966). (In his original paper, the definition of the matrix  $S$  was  $S = [T^{-1}WQ : TKR]$ . However, if we use it, we can not get the result. It should be  $S = [T^{-1}WQ : T'KR]$ ). On the other hand,

$$\begin{aligned} X(X'V^{-1}X)^{-1}X' &= [W : WF] \begin{bmatrix} W'V^{-1}W & : & W'V^{-1}WF \\ F'W'V^{-1}W & : & F'W'V^{-1}WF \end{bmatrix}^{-1} \begin{bmatrix} W' \\ F'W' \end{bmatrix} \\ &= [W : WF] \begin{bmatrix} (W'V^{-1}W)^{-1} & : & \mathbf{0} \\ \mathbf{0} & : & \mathbf{0} \end{bmatrix} \begin{bmatrix} W' \\ F'W' \end{bmatrix} \\ &= W(W'V^{-1}W)^{-1}W'. \end{aligned} \quad (A2)$$

This is always true for any generalized inverse of  $X'V^{-1}X$ , because  $X(X'V^{-1}X)^{-1}X'$  is invariant to the choice of the generalized inverse. Substituting (A2) into (A1), we get :

$$V^{-1} - V^{-1}X(X'V^{-1}X)^{-1}X'V^{-1} = K(K'VK)^{-1}K' \quad (A3)$$

For more detailed discussions, see SEARLE (1979).